

Permutation Reconstruction from *MinMax*-Betweenness Constraints

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Abstract

In this paper, we investigate the reconstruction of permutations on $\{1, 2, \dots, n\}$ from betweenness constraints involving the minimum and the maximum element located between t and $t + 1$, for all $t = 1, 2, \dots, n - 1$. We propose two variants of the problem (directed and undirected), and focus first on the directed version, for which we draw up general features and design a polynomial algorithm in a particular case. Then, we investigate necessary and sufficient conditions for the uniqueness of the reconstruction in both directed and undirected versions, using a parameter k whose variation controls the stringency of the betweenness constraints. We finally point out open problems.

Keywords: betweenness, permutation, algorithm, genome, common intervals

1 Introduction

The BETWEENNESS problem is motivated by physical mapping in molecular biology and the design of circuits [2]. In this problem, we are given the set $[n] := \{1, 2, \dots, n\}$, for some positive integer n , and a set of m *betweenness constraints* ($m > 0$), each represented as a triple $x \xleftrightarrow{a} y$ with $x, a, y \in [n]$ and signifying that a is required to be between x and y . The goal is to find a permutation on $[n]$ satisfying a maximum number of betweenness constraints. In [2], it is shown that the BETWEENNESS problem is NP-complete even in the particular case where all the constraints have to be satisfied.

In this paper we are interested in a problem related to the BETWEENNESS problem, which also finds its motivations in molecular biology. Given K ($K \geq 2$) permutations on the same set $[n]$, representing K genomes given by the sequences of their genes, a *common interval* of these permutations is a subset of $[n]$ whose elements are consecutive (*i.e.* they form an interval) on each of the K permutations. Common intervals thus represent regions of the genomes which have identical gene content, but possibly different gene order. Computing common intervals or specific subclasses of them in linear time (up to the number of output intervals) has been done by case-by-case approaches until recently, when we proposed [3] a common linear framework, whose basis is the notion of *MinMax*-profile. The *MinMax*-profile of a permutation P forgets the order of the elements in a permutation, and keeps only essential betweenness information, defined as, for each $t \in [n - 1]$, the minimum and maximum value in the interval delimited by the elements t (included) and $t + 1$ (included) on P (with no restriction on the relative positions of t and $t + 1$ on P). When K permutations are available, their *MinMax*-profile is defined similarly, by considering for every $t \in [n - 1]$ the global minimum and the global maximum of the K intervals delimited by t and $t + 1$ on the K permutations. We show in [3] that, assuming the permutations have been renumbered such that one of them is the identity permutation, the *MinMax*-profile of K permutations is all we need to find common intervals, as well as all the specific subclasses of common intervals defined in the literature, in linear time (up to the number of output intervals).

Hence, the *MinMax*-profile is a simplified representation of a (set of) permutation(s), which is sufficient to efficiently solve a number of problems related to finding common intervals in permutations. Moreover, it may be computed in linear time [3]. However, it can be easily seen that distinct (sets of) permutations may have the same *MinMax*-profile, implying that the *MinMax*-profile captures a part, but not all, of the information in the (set of) permutation(s).

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In this paper, we study the reconstruction of a permutation from a given *MinMax*-profile, and discuss possible generalizations.

2 Definitions and Problems

In the remaining of the paper, permutations are defined on $[n]$ and are increased with elements 0 and $n + 1$, added respectively at the beginning and the end of each permutation (and assumed to be fixed). This is due to the need to make the distinction between a permutation and its reverse order permutation.

Definition 1. [3] The *MinMax-profile* of a permutation P on $[n] \cup \{0, n + 1\}$ is the set of *MinMax-constraints*

$$\text{MinMax}(P) = \{t \xrightarrow{[\min_t, \max_t]} t + 1 \mid 0 \leq t \leq n\}$$

where \min_t (\max_t respectively) is the minimum (maximum respectively) element in the interval delimited on P by the element t (included) and the element $t + 1$ (included).

Note that the relative positions on P (i.e. which one is on the left of the other) of $t, t + 1$ on the one hand, and of \min_t, \max_t on the other hand are not indicated by a *MinMax*-profile. In the case where the relative positions of t and $t + 1$ are known for all t , we use the term of *directed MinMax-profile* and the notations $t \xrightarrow{[\min_t, \max_t]} t + 1$ when t is on the left of $t + 1$, respectively $t \xleftarrow{[\min_t, \max_t]} t + 1$ when $t + 1$ is on the left of t .

Example 1. Let $P = (0\ 6\ 4\ 7\ 2\ 9\ 1\ 8\ 5\ 3\ 10)$ a permutation on $[9] \cup \{0, 10\}$. Then its *MinMax*-profile is (note that the *MinMax*-constraints sharing an element are concatenated):

$$0 \xrightarrow{[0,9]} 1 \xrightarrow{[1,9]} 2 \xrightarrow{[1,9]} 3 \xrightarrow{[1,9]} 4 \xrightarrow{[1,9]} 5 \xrightarrow{[1,9]} 6 \xrightarrow{[4,7]} 7 \xrightarrow{[1,9]} 8 \xrightarrow{[1,9]} 9 \xrightarrow{[1,10]} 10$$

whereas its directed *MinMax*-profile is:

$$0 \xrightarrow{[0,9]} 1 \xleftarrow{[1,9]} 2 \xrightarrow{[1,9]} 3 \xleftarrow{[1,9]} 4 \xrightarrow{[1,9]} 5 \xleftarrow{[1,9]} 6 \xrightarrow{[4,7]} 7 \xrightarrow{[1,9]} 8 \xleftarrow{[1,9]} 9 \xrightarrow{[1,10]} 10$$

Notice that the *MinMax*-profile and the directed *MinMax*-profile of any permutation obtained by arbitrarily permuting the elements $\{3, 5, 8\}$ are the same, showing that a (directed or not) *MinMax*-profile may correspond to several distinct permutations.

The *MinMax-profile* of a set \mathcal{P} of permutations is defined similarly [3], by requiring that \min_t and \max_t be defined over the union of the intervals delimited by t (included) and $t + 1$ (included) on the K permutations in \mathcal{P} . This definition is given here for the sake of completeness, but is little used in the paper.

We distinguish between the *MinMax*-profile of a (set of) permutation(s) and a *MinMax-profile*:

Definition 2. A *MinMax-profile* on $[n] \cup \{0, n + 1\}$ is a set of *MinMax-constraints*

$$F = \{t \xrightarrow{[m_t, M_t]} t + 1 \mid 0 \leq t \leq n\}$$

with $0 \leq m_t \leq t < t + 1 \leq M_t \leq n + 1$.

Again, a *MinMax-profile* is *directed* when for all t , $0 \leq t \leq n$, the relative position of t with respect to $t + 1$ is given. A *MinMax-profile* may be the *MinMax*-profile of some permutation, or of a set of permutations, but may also be the profile of no (set of) permutation(s). We limit this study to one permutation, and therefore formulate the following problem:

MinMax-BETWEENNESS

Input: A positive integer n , a *MinMax*-profile F on $[n] \cup \{0, n+1\}$.

Question: Is there a permutation P on $[n] \cup \{0, n+1\}$ whose *MinMax*-profile is F ?

The *MinMax*-BETWEENNESS problem is obviously related to the BETWEENNESS problem, since looking for a permutation P with *MinMax*-constraints defined by F means satisfying a number of betweenness constraints. Some differences exist however, as F also defines non-betweenness constraints. More precisely, each *MinMax*-constraint $t \xrightarrow{m_t, M_t} t+1$ from F may be expressed using the betweenness constraints (abbreviated B-constraints):

$$t \xleftrightarrow{m_t} t+1, t \xleftrightarrow{M_t} t+1 \quad (1)$$

along with the non-betweenness constraints (abbreviated NB-constraints):

$$\neg(t \xleftrightarrow{j} t+1), j = 0, 1, \dots, m_t - 1, M_t + 1, \dots, n+1. \quad (2)$$

It is easy to imagine that in the *MinMax*-BETWEENNESS problem, the lack of information about the relative position of t and $t+1$ on the permutation P (i.e. which one is on the left of the other) is a major difficulty. The directed version of the problem, in which these relative positions are given, should possibly be easier.

DIRECTED *MinMax*-BETWEENNESS

Input: A positive integer n , a directed *MinMax*-profile F on $[n] \cup \{0, n+1\}$.

Question: Is there a permutation P on $[n] \cup \{0, n+1\}$ whose directed *MinMax*-profile is F ?

Remark 1. It is worth noticing here that in a (directed or not) *MinMax*-profile which corresponds to at least one permutation on $[n] \cup \{0, n+1\}$, the value 0 ($n+1$ respectively) should only occur in one precise *MinMax*-constraint, namely the one involving 0 and 1 (n and $n+1$ respectively). Otherwise, 0 ($n+1$ respectively) cannot be the leftmost (rightmost, respectively) value in the permutation. In the subsequent of the paper, it is assumed that this condition has been verified before further investigations, and assume therefore that 0 and $n+1$ are respectively located in places 0 and $n+1$.

We present below, in Section 3, our analysis of the Directed *MinMax*-BETWEENNESS problem, proposing a first algorithmic approach and pointing out the main difficulties for reaching a complete polynomial solution. In Section 4, we identify a polynomial particular case for the directed version. In Section 5 we propose to generalize *MinMax*-profiles to k -profiles, by introducing a parameter k which allows to progressively increase the amount of information contained in a k -profile, up to a value k_0 which allows to identify each permutation by its k_0 -profile. Section 6 is the conclusion.

3 Seeking an algorithm for DIRECTED *MinMax*-BETWEENNESS

3.1 A naïve approach

Let F be a directed *MinMax*-profile on $[n] \cup \{0, n+1\}$. The most intuitive idea for solving Directed *MinMax*-BETWEENNESS is to build a simple directed graph G (i.e. with no loops or multiple arcs) whose vertex set $V(G)$ is $[n] \cup \{0, n+1\}$ and whose arcs (x, y) indicate the precedence relationships between the elements on each permutation corresponding to the given k -profile (i.e. x is on the left of y). If a permutation exists, G must be a directed acyclic graph (or DAG). The *MinMax*-constraints from F directly define arcs using: 1) the relative order between t and $t+1$, for each $t \in [n]$ (the corresponding arcs of G are called *R-arcs*), and 2) the B-constraints (resulting into *B-arcs*). Further arcs may be dynamically obtained by repeatedly invoking: 3) the transitivity of the precedence relationship (resulting into *T-arcs*), and 4) the NB-constraints (resulting into *NB-arcs*).

Algorithm 1 The Build-Easy-Arcs algorithm

Input: A directed *MinMax*-profile F over $[n] \cup \{0, n+1\}$.

Output: Either the answer "No" (meaning no permutation exists), or the pair (G, SilNB) where G is the DAG containing all deducible R -, N -, T - and NB -arcs, and SilNB is the set of silent NB-constraints.

(Note: Arcs are added only if they do not create loops, nor multiple arcs with common source and target.)

```
1:  $G \leftarrow ([n] \cup \{0, n+1\}, \emptyset)$ 
2: for each  $t \in [n] \cup \{0\}$  do
3:   if  $t \xrightarrow{[m_t, M_t]} t+1 \in F$  then  $tl \leftarrow t; tr \leftarrow t+1$  else  $tl \leftarrow t+1; tr \leftarrow t$  end if
4:   add the  $R$ -arc  $(tl, tr)$  to  $G$ 
5:   add the  $B$ -arcs  $(tl, m_t), (m_t, tr), (tl, M_t), (M_t, tr)$  to  $G$  // according to (1)
6: end for
7:  $\text{SilNB} \leftarrow$  the set of all NB-constraints  $\neg(t \xleftrightarrow{a} t+1)$  deduced from  $F$  // according to (2)
8:  $G \leftarrow \text{Build-Closure}(G, \text{SilNB})$ 
9: remove from  $\text{SilNB}$  all NB-constraints  $\neg(t \xleftrightarrow{a} t+1)$  for which a setting is already found
10: if  $G$  is not a DAG then
11:   output "No"
12: else
13:   output  $(G, \text{SilNB})$ 
14: end if
```

Algorithm 2 The Build-Closure algorithm

Input: A simple directed graph G with vertex set $[n] \cup \{0, n+1\}$, a set NBc of NB-constraints on $[n] \cup \{0, n+1\}$.

Output: The NB-transitive closure of G using the NB-constraints in NBc .

(Note: Arcs are added only if they do not create loops, nor multiple arcs with common source and target.)

```
1: while a  $T$ -arc or an  $NB$ -arc  $(x, y)$  may be added do
2:   add  $(x, y)$  to  $G$ 
3: end while
4: output( $G$ )
```

Algorithm 1 shows these steps. After the construction of the R - and B -arcs (steps 2-6), either transitivity or NB-constraints may be arbitrarily invoked to add supplementary arcs as long as possible, performing what we call the *NB-transitive closure* of G . This is done by the Build-Closure algorithm (Algorithm 2), called in step 8 of Algorithm 1. It is clear that in step 1 of the Build-Closure algorithm a T -arc (x, y) may be added iff there is a vertex c such that (x, c) and (c, y) are arcs, but (x, y) is not an arc. The condition for adding the NB -arc (x, y) is slightly more complex, as (x, y) may be added iff

- either an NB-constraint $\neg(y \xleftrightarrow{x} z)$ with $z \in \{y-1, y+1\}$ exists in NBc , and (x, z) is an arc,
- or an NB-constraint $\neg(x \xleftrightarrow{y} z)$ with $z \in \{x-1, x+1\}$ exists in NBc , and (z, y) is an arc.

Clearly, this naïve approach for *MinMax*-BETWEENNESS attempts to exploit all the *MinMax*-constraints. Unfortunately, for some NB-constraints $\neg(t \xleftrightarrow{a} t+1)$ Algorithm 1 may provide no setting (*i.e.* neither the arcs (t, a) and $(t+1, a)$, nor the arcs (a, t) and $(a, t+1)$ are present in G), as shown below. These constraints are called *silent NB-constraints*, and are returned by the algorithm together with G , if G is a DAG (step 13).

Example 2. Let F be defined on $[9] \cup \{0, 10\}$ by the following *MinMax*-constraints:

$$0 \xrightarrow{[0,9]} 1 \xleftarrow{[1,9]} 2 \xrightarrow{[1,9]} 3 \xleftarrow{[1,9]} 4 \xrightarrow{[1,9]} 5 \xleftarrow{[1,9]} 6 \xrightarrow{[4,7]} 7 \xrightarrow{[1,9]} 8 \xleftarrow{[1,9]} 9 \xrightarrow{[1,10]} 10$$

Figure 1 shows the R -, B - and T - arcs used by Algorithm 1 to build the directed graph deduced from F . Vertices 0 and 10 are left apart in this figure, since the constraints they are involved in allow only to

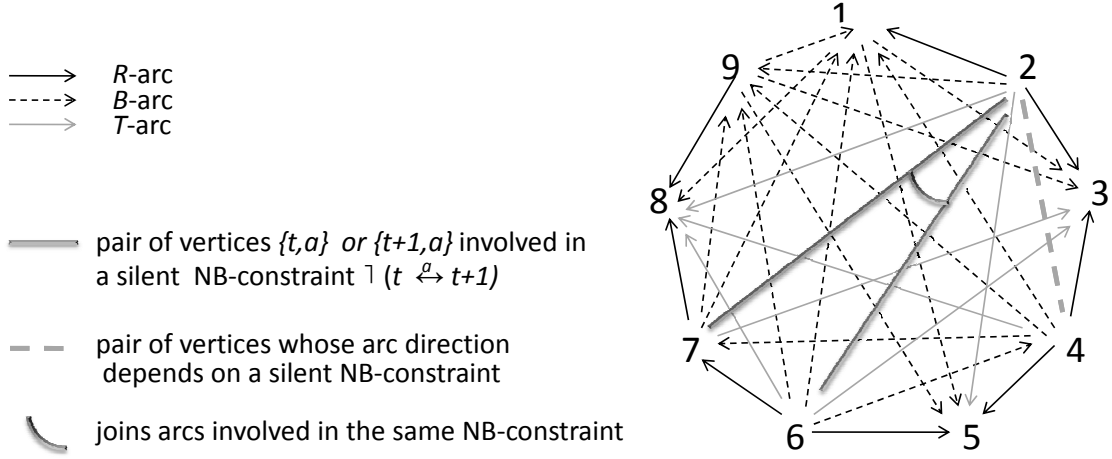


Figure 1: Directed acyclic graph G obtained from Algorithm 1 using the directed *MinMax*-profile $0 \xrightarrow{[0,9]} 1 \xleftarrow{[1,9]} 2 \xrightarrow{[1,9]} 3 \xleftarrow{[1,9]} 4 \xrightarrow{[1,9]} 5 \xleftarrow{[1,9]} 6 \xrightarrow{[4,7]} 7 \xrightarrow{[1,9]} 8 \xleftarrow{[1,9]} 9 \xrightarrow{[1,10]} 10$. For simplicity reasons, vertices 0 and 10 and the arcs incident to at least one of them are omitted. The pairs of vertices $\{2, 4\}$, $\{2, 6\}$ and $\{2, 7\}$ are not arcs, they show the silent NB-constraint $\neg(6 \xrightarrow{2} 7)$ and the pair $\{2, 4\}$, that are related by coherent arc directions.

place them at the beginning and respectively at the end of the sought permutations. The NB-constraints imposed by F (except those with 0 and 10) are $6 \xrightarrow{u} 7$ with $u \in \{1, 2, 3, 8, 9\}$. When $u = 1$ and $u = 9$, both arcs involved in the NB-constraint are already in G (due to B -constraints). For $u = 3$ and $u = 8$, all the arcs with 6 and 7 are built by transitivity (although some of them may also be built using the appropriate NB-constraints), during the steps 8 in Algorithm 1. For $u = 2$, the NB-constraint cannot be used, since none of the arcs exists (and no other arc may be created by transitivity). Then we have $SilNB = \{\neg(6 \xrightarrow{2} 7)\}$ at the end of Algorithm 1. Notice that the pairs $\{2, 4\}$, $\{2, 6\}$ and $\{2, 7\}$ have correlated directions in any setting, that is, either all three arcs have the source 2, or all three arcs have the target 2. For $\{2, 6\}$ and $\{2, 7\}$ this is due to the NB-constraint $\neg(6 \xrightarrow{2} 7)$, whereas for $\{2, 4\}$ this is due to the transitivity ensured by the arcs $(6, 4)$ and $(4, 7)$.

Our problem is now this one:

(P) Given G and a set of silent NB-constraints, decide whether a setting is possible for each silent NB-constraint such that the graph resulting by transitive closure is a DAG.

Unfortunately, the following result shows the difficulty of the problem:

Claim 1. [1] Problem (P) is NP-complete even when the silent NB-constraints involve disjoint triples of vertices.

Notice however that the graph G we obtain at the end of Algorithm 1 may have particular features (that we have not identified) making that we are dealing with a particular case of problem (P). Claim 1 shows therefore that our problem is potentially difficult, but does not prove its hardness.

Remark 2. From an algorithmic point of view, we may notice that with the output of Algorithm 1 we may easily find a parameterized algorithm for *MinMax*-BETWEENNESS. Given G and $SilNB$, we have $O(2^{|SilNB|})$ possible settings to test, thus resulting into an FPT algorithm with parameter s given by the number of silent NB-constraints.

3.2 Further analysis of arc propagation

With the aim of forcing the setting of some appropriately chosen silent NB-constraint, let us now analyze the impact of adding an arbitrary arc (a_1, b_1) to G , where a_1 and b_1 are non-adjacent vertices from G . Denote $G + (a_1, b_1)$ the graph obtained from G by adding the arc (a_1, b_1) , and let G^1 be the NB-transitive closure of $(G + (a_1, b_1), \text{SilNB})$, i.e. the directed graph obtained by performing Build-Closure($G + (a_1, b_1), \text{SilNB}$).

Several definitions are needed before going further. Given an NB-constraint $\neg(t \xrightarrow{a} t+1)$, the vertex a of G is called the *top* of the NB-constraint, whereas the pair $\{t, t+1\}$ is called the *basis* of the NB-constraint. An arc (x, y) is *new* if it is an arc of G^1 but not of G , and is *old* if it is an arc of G . New arcs are obtained using Build-Closure($G + (a_1, b_1), \text{SilNB}$) according to a certain linear order, resulting from the arbitrary choices made in step 1. This order is denoted α , such that $(x_1, y_1) \alpha (x_2, y_2)$ means that (x_1, y_1) is created by Build-Closure before (x_2, y_2) . Then, the following claim is simple:

Claim 2. *For each new arc (v, w) , there exists a series of new arcs $U := (v_1, w_1), (v_2, w_2), \dots, (v_z, w_z)$ such that $(v_1, w_1) = (a_1, b_1)$, $(v_z, w_z) = (v, w)$, $(v_i, w_i) \alpha (v_{i+1}, w_{i+1})$ for all i with $1 \leq i \leq z-1$ and each arc (v_{i+1}, w_{i+1}) , $1 \leq i \leq z-1$, is obtained from the preceding one (v_i, w_i) using one of the following cases:*

1. $w_{i+1} = w_i$ and (v_{i+1}, v_i) is either an old arc, or a new arc such that $(v_{i+1}, v_i) \alpha (v_{i+1}, w_{i+1})$; in this case (v_{i+1}, w_{i+1}) is a new *T*-arc.
2. $w_i = w_{i+1}$ and $\{v_i, v_{i+1}\}$ is the basis of an NB-constraint of SilNB with top w_i ; in this case (v_{i+1}, w_{i+1}) is a new *NB*-arc.
3. $v_i = v_{i+1}$ and (w_i, w_{i+1}) is either an old arc, or a new arc such that $(w_i, w_{i+1}) \alpha (v_{i+1}, w_{i+1})$; in this case (v_{i+1}, w_{i+1}) is a new *T*-arc.
4. $v_i = v_{i+1}$ and $\{w_i, w_{i+1}\}$ is the basis of an NB-constraint from SilNB with top v_i ; in this case (v_{i+1}, w_{i+1}) is a new *NB*-arc.

Proof. In order to obtain (v_{i+1}, w_{i+1}) , we need to apply either the transitivity (step 2 in Algorithm 2 for a *T*-arc, which gives cases 1 and 3), or an NB-constraint from SilNB (again step 2 in Algorithm 2, but for an *NB*-arc, which gives cases 2 and 4). ■

The sequence U is called a *setting sequence* for (v, w) , whereas the index i of an arc (v_i, w_i) is called its *range* in U . From now on, the case in Claim 2 used to deduce one arc from the preceding one in a setting sequence is indicated between the two arcs.

Example 3. For the example in Figure 1, if $a_1 = 2$ and $b_1 = 7$, then $U := (2, 7) \xrightarrow{4} (2, 6) \xrightarrow{3} (2, 4)$ is a setting sequence for $(2, 4)$ using case 4 followed by case 3 in Claim 2 to go from one arc to the next one.

Now, let a_1, a_2, \dots, a_s (respectively b_1, b_2, \dots, b_r) be the subsequence of v_1, \dots, v_z (respectively of w_1, \dots, w_z) obtained by replacing *consecutive* copies of the same vertex with only one copy of that vertex. Equivalently, if (a_i, b_j) is an arc of U , then the next arc is either (a_{i+1}, b_j) (cases 1 and 2 in Claim 2) or (a_i, b_{j+1}) (cases 3 and 4 in Claim 2). Of course, we have $a_1 = v_1, a_s = v_z = v, b_1 = w_1$ and $b_r = w_z = w$.

Example 4. Consider $P = (0\ 7\ 4\ 10\ 2\ 1\ 12\ 8\ 3\ 9\ 5\ 11\ 6\ 13)$, and let F be the *MinMax*-profile of P . For F , apply Algorithm 1 to obtain the graph G and the set SilNB . Then G - that the reader is invited to build it himself - is partitioned into three sets, respectively made of: the vertices preceding the pair $\{1, 12\}$, the pair $\{1, 12\}$ (in this order, and with no intermediate vertex), and the vertices following the pair $\{1, 12\}$. The set SilNB is $\{8 \xrightarrow{11} 9, 5 \xrightarrow{3} 6\}$, and thus involves only vertices in the third set, which induces in G the subgraph G' with vertex set $\{3, 5, 6, 8, 9, 11\}$ and arcs

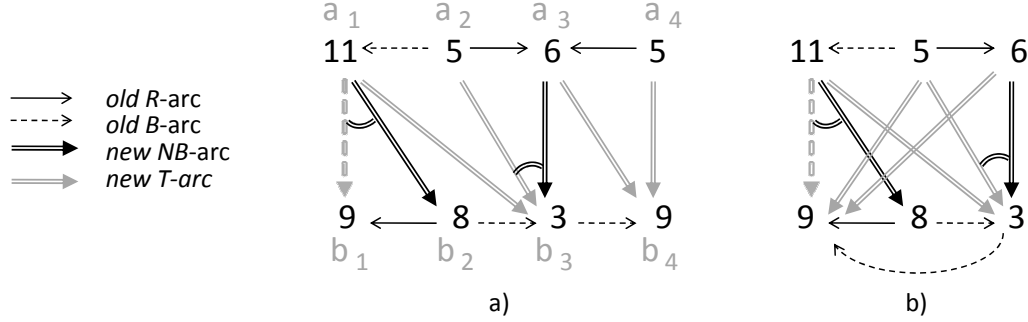


Figure 2: Old arcs (thin horizontal arrows) and new arcs (thick vertical arrows) used by the setting sequence $U := (11, 9)^4(11, 8)^3(11, 3)^1(5, 3)^2(6, 3)^3(6, 9)^1(5, 9)$ in Example 4. a) Using notation a_i , $1 \leq i \leq 4$, and b_j , $1 \leq j \leq 4$. b) Using the vertices in G , and thus defining the graph H_U . In both cases, the initial arc (a_1, b_1) is a dotted arrow.

$\{(8, 3), (3, 9), (8, 9)\} \cup \{(5, 11), (11, 6), (5, 6)\}$. With $(a_1, b_1) = (11, 9)$, we have (see Figure 2a) that $U := (11, 9)^4(11, 8)^3(11, 3)^1(5, 3)^2(6, 3)^3(6, 9)^1(5, 9)$ is a setting sequence for $(5, 9)$ with $a_1 = 11, a_2 = 5, a_3 = 6, a_4 = 5$ (thus $s = 4$) and $b_1 = 9, b_2 = 8, b_3 = 3, b_4 = 9$ (thus $r = 4$).

Remark 3. Notice that we could possibly have $a_i = a_l$, for distinct $i, l \in \{1, 2, \dots, s\}$, i.e. they correspond to the same vertex of G , if two arcs with the same endpoint are set in distant steps of the setting process represented by U . We could also possibly have $a_i = b_j$ for some i, j if, for instance, a_1, \dots, a_h (with $h > i$) are distinct, b_1, \dots, b_{j-1} are distinct, (a_h, b_{j-1}) is a new arc and (b_{j-1}, a_i) is an old arc (making that the vertex b_j is equal to a_i , and thus by transitivity - or case 3 in Claim 2 - one sets (a_h, a_i)).

Remark 4. Also note that for every pair of arcs (a_i, b_j) and (a_p, b_q) from U , we have either $i \leq p$ and $j \leq q$ (when $(a_i, b_j) \alpha (a_p, b_q)$), or $i \geq p$ and $j \geq q$ (when $(a_p, b_q) \alpha (a_i, b_j)$). It is therefore understood, here and in the subsequent of the paper, that in case $a_i = a_l$ for some $i \neq l$, we make a clear difference between the arcs (a_i, b_j) of U and the arcs (a_l, b_f) of U . These arcs are incident with the same vertex of G but this vertex is called a_i in the first case, and a_l in the second one.

Example 4 (cont'd). We have $a_2 = a_4 = 5$ and $b_1 = b_4 = 9$, but when we refer to the new arcs containing a_2 we only refer to the arc $(5, 3)$ and when we refer to the new arcs containing a_4 we only refer to the arc $(5, 9)$. Similarly, when we refer to the new arcs incident with b_1 we refer only to the arc $(11, 9)$ whereas when we refer to those incident with b_4 we mean the arcs $(6, 9)$ and $(5, 9)$.

In order to represent arc propagation, we need to look closer to the partial subgraph H_U of G^1 given by the set of *distinct* vertices used in the setting sequence U , the arcs in U and the arcs used to deduce each arc of U from the previous one, using Claim 2. The graph H_U is defined as:

$$\begin{aligned}
 V(H_U) &= \{a_i \in V(G) \mid 1 \leq i \leq s\} \cup \{b_j \in V(G) \mid 1 \leq j \leq r\} \\
 E(H_U) &= U \cup \{(a_{i+1}, a_i) \mid \exists b_j : (a_{i+1}, b_j) \text{ is deduced from } (a_i, b_j) \text{ in } U \text{ using case 1}\} \\
 &\quad \cup \{(a_i, a_{i+1}) \mid \exists b_j : (a_{i+1}, b_j) \text{ is deduced from } (a_i, b_j) \text{ in } U \text{ using case 2}\} \\
 &\quad \cup \{(b_j, b_{j+1}) \mid \exists a_i : (a_i, b_{j+1}) \text{ is deduced from } (a_i, b_j) \text{ in } U \text{ using case 3}\} \\
 &\quad \cup \{(b_{j+1}, b_j) \mid \exists a_i : (a_i, b_{j+1}) \text{ is deduced from } (a_i, b_j) \text{ in } U \text{ using case 4}\}
 \end{aligned}$$

The graph H_U is called the *setting path* associated with U (or, alternatively, a *setting path* for (a_s, b_r)). Notice that case 2 (respectively case 4) in Claim 2 may be included in case 1 (respectively case 3)

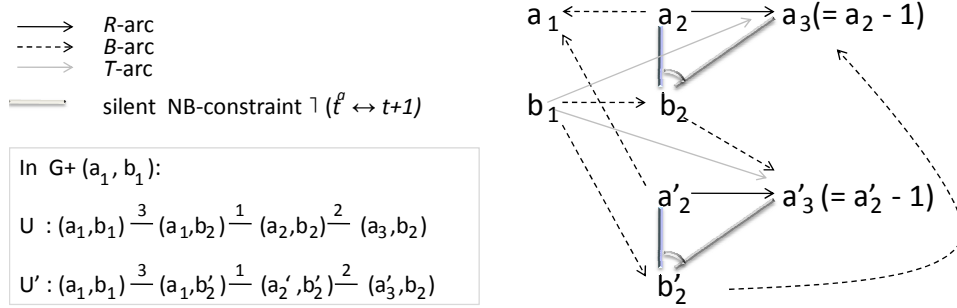
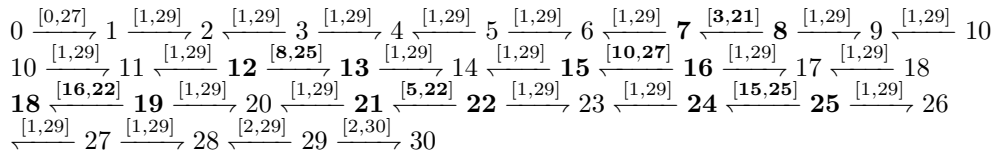


Figure 3: Configuration where the addition of the arc (a_1, b_1) sets (among other arcs) the arcs (a_3, b_2) and (a'_3, b'_2) , thus inducing a circuit. The types of the arcs are the same as in Figure 1, but the pairs depending on the setting of a silent NB-constraint are missing.

when the basis is the arc (a_{i+1}, a_i) (arc (b_j, b_{j+1}) respectively). The definition of H_U keeps as case 2 (respectively case 4) only the configuration not included in case 1 (respectively case 3). See Figure 2b).

Claim 2 and the definition of H_U allow us to have a basis for future analysis, but also show us that the choice of one arc (a_1, b_1) has effects that are difficult to measure accurately. The NP-completeness of the problem **(P)** (see Claim 1) comes from this complex arc propagation, which makes that different setting sequences with the same initial arc may lead to conflicts, *i.e.* to circuits.

Example 5. In Figure 3, we present a configuration (which is a subgraph of G) showing that not each possible setting is a correct setting, since imposing the existence of one arc (a_1, b_1) may induce circuits in the graph G^1 . In this configuration, setting the arc (a_1, b_1) implies the additional arcs (a_3, b_2) and (a'_3, b'_2) , and thus the construction of a circuit. A *MinMax*-profile inducing such a configuration in the associated DAG G is the following one (where $a_1 = 18, b_1 = 12, a_2 = 22, a_3 = 21, a'_2 = 16, a'_3 = 15, b_2 = 25, b'_2 = 8$):



In this example, the *MinMax*-constraints in bold define the arcs needed to obtain the configuration in Figure 3, and some additional arcs. The elements involved in these *MinMax*-constraints are, in every permutation with this *MinMax*-profile, on the left of 1 and 29 (the minimum and maximum elements), which are neighbors and in this order on each permutation. The remaining of the elements are on the right of 1 and 29, and are intended to complete the set $\{1, 2, \dots, 30\}$ without any participation to the configuration.

In order to find polynomial particular cases, we need to be able to control the form of the setting paths, and this is what we do in the subsequent. To this end, notice that:

Remark 5. The vertices 0 and $n + 1$ belong to no setting path. Indeed, according to Remark 1, it is assumed that they are definitely located at places 0 and $n + 1$ respectively, and thus their relative positions with respect to any other element are known. No arc incident to any of them may thus be added, as would be the case if they belonged to some setting path.

4 Polynomial case for DIRECTED *MinMax*-BETWEENNESS

Say that a *MinMax*-profile F on $[n] \cup \{0, n+1\}$ is *linear* if the inclusion between sets defines a linear order on the intervals $[m_t..M_t]$, $1 \leq t \leq n-1$, where the notation $(a..b)$ denotes the set of integers x with $a \leq x \leq b$. We show in this section that the problem DIRECTED *MinMax*-BETWEENNESS is polynomial for linear *MinMax*-profiles.

Given $c \in [n]$, let $NB(c) = \{t \mid 1 \leq t \leq n-1, c < m_t \text{ or } M_t < c\}$. In other words, $NB(c)$ is the set of values t such that $\{t, t+1\}$ is the basis of an NB-constraint with top c .

Claim 3. *Let F be a linear profile on $[n] \cup \{0, n+1\}$. Then the inclusion between sets defines a linear order denoted \prec on the sets $NB(c)$, $1 \leq c \leq n$.*

Proof. By contradiction, assume that c_1 and c_2 exist such that $NB(c_1) \setminus NB(c_2)$ contains t_1 and $NB(c_2) \setminus NB(c_1)$ contains t_2 . Then $t_1 \neq t_2$.

In the case where $c_1 < m_{t_1}$ and $c_2 < m_{t_2}$, assume w.l.o.g. that $c_1 < c_2$. Then $c_1 < m_{t_2}$ and thus $t_2 \in NB(c_1)$, a contradiction. The case where $c_1 > M_{t_1}$ and $c_2 > M_{t_2}$ is similar.

In the case where $c_1 < m_{t_1}$ and $c_2 > M_{t_2}$, recall that by hypothesis F is linear, and thus either $[m_{t_1}..M_{t_1}] \subseteq [m_{t_2}..M_{t_2}]$ or vice-versa. If $[m_{t_1}..M_{t_1}] \subseteq [m_{t_2}..M_{t_2}]$, then $m_{t_2} \leq m_{t_1} < M_{t_1} \leq M_{t_2}$ and with $c_2 > M_{t_2}$ we deduce that $t_1 \in NB(c_2)$, a contradiction. If $[m_{t_2}..M_{t_2}] \subseteq [m_{t_1}..M_{t_1}]$, then $m_{t_1} \leq m_{t_2} < M_{t_2} \leq M_{t_1}$ and with $c_1 < m_{t_1}$ we deduce that $t_2 \in NB(c_1)$, a contradiction. ■

Now, assume Algorithm 1 has been applied for F , and let (G, SilNB) be its output, assuming G is a DAG. To finish the algorithm for F , we apply Algorithm 3. The following claim is easy but very useful.

Claim 4. *The vertex b_1 chosen in Algorithm 3 has the following properties:*

- (a) $NB(b_1)$ is maximum with respect to the linear order \prec on the set $\{NB(c) \mid 1 \leq c \leq n \text{ and } c \text{ is the top of at least one constraint from SilNB}\}$.
- (b) b_1 does not belong to a basis, but is a top for all the basis defining constraints from SilNB .

Proof. The first affirmation is clear by the choice of b_1 in step 3 of the algorithm and Claim 3. The second affirmation is deduced by contradiction. If b_1 belonged to a basis $\{b_1, b_1+1\}$ or $\{b_1-1, b_1\}$ with top c , then we would have $NB(c) \not\subseteq NB(b_1)$ since the basis $\{b_1, b_1+1\}$ or $\{b_1-1, b_1\}$ cannot have top b_1 (the vertices of a basis are by definition distinct from its top). The second part of affirmation (b) results directly from affirmation (a). ■

In the next claims, we show the correctness of our algorithm. To this end, each arc (a_i, b_j) of U (and thus of H_U) is called a *local new arc* with respect to U , in order to make the difference with the arcs from G^1 which are new but do not belong to U , termed *non-local new arcs*. Similarly, a vertex a_i of H_U is a *local top* if there exists b_q such that $\neg(b_q \xrightarrow{a_i} b_{q+1})$ is an NB-constraint used by U , i.e. one of the arcs (a_i, b_q) and (a_i, b_{q+1}) is deduced from the other in H_U , using case 4. The pair $\{b_q, b_{q+1}\}$ is in this case a *local basis*. The symmetric definitions hold for a vertex b_j (instead of a_i). Note that a local basis has a unique local top, by Remark 4.

For any vertex a_i , we also denote $\text{first}_U(i)$ the minimum u with $1 \leq u \leq r$ such that (a_i, b_u) belongs to U .

Claim 5. *Let F be a directed linear profile on $[n] \cup \{0, n+1\}$ and let a_1 and b_1 be chosen as in Algorithm 3. Then the following affirmations hold:*

- (a) *Let (v, w) be a new arc of G^1 and let U be setting sequence for (v, w) with arc sources a_1, a_2, \dots, a_s and arc targets b_1, b_2, \dots, b_r . Then there is no old arc (b_1, a_i) in G^1 , with $1 < i \leq s$.*
- (b) *All arcs (b_1, x) of G^1 are old.*

Algorithm 3 The Linear-Profile algorithm

Input: The output $(G, SilNB)$ of Algorithm 1 for a directed linear *MinMax*-profile F on $[n] \cup \{0, n+1\}$.

Output: A permutation P with the *MinMax*-profile F .

- 1: **while** $SilNB \neq \emptyset$ **do**
 - 2: $C \leftarrow \{c \in [n] \mid c \text{ is the top of at least one constraint from } SilNB\}$
 - 3: Choose $b_1 \in C$ s.t. $|NB(b_1)| = \max\{|NB(c)| \mid c \in C\}$
 - 4: Choose a_1 such that $\neg(a_1 \xrightarrow{b_1} a_1 + 1) \in SilNB$.
 - 5: $G \leftarrow \text{Build-Closure}(G + (a_1, b_1), SilNB)$
 - 6: **end while**
 - 7: $P \leftarrow$ topologically sort G
 - 8: **Output**(P)
-

Proof. To prove (a), we assume by contradiction that the affirmation is false, and choose $(v, w), U$ and (b_1, a_i) such that the arc $(a_i, b_{first_U(i)})$ is the smallest with respect to the order α . Several cases occur.

- i) If $\{a_{i-1}, a_i\}$ is a local basis (case 2 in the definition of H_U), then b_1 is also a top of it (by Claim 4(b) and thus from the old arc (b_1, a_i) we deduce the existence of the old arc (b_1, a_{i-1}) (computed by the call of Build-Closure in step 8 of Algorithm 1). But then the choice of (b_1, a_i) is contradicted, since $(a_{i-1}, b_{first_U(i-1)})\alpha(a_i, b_{first_U(i)})$.
- ii) If (a_i, a_{i-1}) is an old arc (case 1 in the definition of H_U , with an old arc), then (b_1, a_{i-1}) is also an old arc, computed by the call of Build-Closure in step 8 of Algorithm 1. As before, the choice of (b_1, a_i) is contradicted.
- iii) If (a_i, a_{i-1}) is a local new arc (case 1 in the definition of H_U , with a local new arc), then this arc belongs to U and was built before $(a_i, b_{first_U(i)})$ since it must be built before its use. Then there exist p, q with $1 \leq p < i - 1$ and $1 \leq q \leq first_U(i)$ such that (a_p, b_q) and (a_i, a_{i-1}) are the same arc, but with different notations due to its multiple use in H_U (see Remark 3). In particular, a_p and a_i are the same vertex of H_U , and thus (b_1, a_p) is an old arc of H_U , with $p < i$. Once again, the choice of (b_1, a_i) is contradicted, since $(a_p, b_{first_U(p)})\alpha(a_i, b_{first_U(i)})$.
- iv) Finally, if (a_i, a_{i-1}) is a non-local new arc (case 1 in the definition of H_U , with a non-local new arc), then it was built before $(a_i, b_{first_U(i)})$. Consequently, there exists a setting sequence T for (a_i, a_{i-1}) with arc sources $c_1(= a_1), c_2, \dots, c_g = a_i$ and arc targets $d_1(= b_1), d_2, \dots, d_h = a_{i-1}$. In this setting sequence, we have that (b_1, c_g) is an old arc, and $(c_g, d_h) = (a_i, a_{i-1})$. Then, $(c_g, d_{first_T(g)})\alpha(a_i, a_{i-1})\alpha(a_i, b_{first_U(i)})$, contradicting again the choice of U and (b_1, a_i) .

To prove (b), assume by contradiction that some arcs (b_1, x) are created by $\text{Build-Closure}(G + \{a_1, b_1\}, SilNB)$, and let (b_1, x_1) be the smallest of them according to the order α . Then in a setting sequence U for (b_1, x_1) with arc sources a_1, a_2, \dots, a_s and arc targets b_1, b_2, \dots, b_r , we have $(b_1, x_1) = (a_p, b_q)$ for some p, q with $1 < p \leq s$ and $1 < q \leq r$. Then the pair $\{a_{p-1}, a_p\}$ is not a basis since $a_p = b_1$ and by Claim 4(b), b_1 belongs to no basis. Then, (a_p, a_{p-1}) is an arc. This arc cannot be old, since then recalling that $a_p = b_1$ we have that (b_1, a_{p-1}) is an old arc thus contradicting affirmation (a). Then (a_p, a_{p-1}) must be a new arc. Now, we have by case 1 in Claim 2 that $(a_p, a_{p-1})\alpha(a_p, b_{first_U(p)})\alpha(a_p, b_q)$. Since $(a_p, a_{p-1}) = (b_1, a_{p-1})$ and $(a_p, b_q) = (b_1, x_1)$ we deduce that $(b_1, a_{p-1})\alpha(b_1, x_1)$, thus contradicting the choice of (b_1, x_1) . ■ Say that a setting sequence U for

(v, w) with arc sources a_1, a_2, \dots, a_s and arc targets b_1, b_2, \dots, b_r is *canonical* if H_U has the following properties:

- (a) b_1 and (if it exists) b_2 are distinct from a_i , $1 \leq i \leq s$, and (b_1, b_2) is an old arc.

(b) $r \leq 2$.

(c) $(a_i, b_1) \in U$, for all i with $1 \leq i \leq s$.

Claim 6. *Let F be a directed linear profile on $[n] \cup \{0, n+1\}$ and let a_1 and b_1 be chosen as in Algorithm 3. Let (v, w) be a new arc of G^1 . Then, for each setting sequence U for (v, w) with arc sources a_1, a_2, \dots, a_s and arc targets b_1, b_2, \dots, b_r , there is a canonical setting sequence U^0 for (v, w) with arc sources a_1, a_2, \dots, a_s and arc targets b_1 , and (whenever $b_1 \neq b_r$) $b'_2 = b_r$.*

Proof. The proof is by induction on the range k of (v, w) (or, equivalently, of (a_s, b_r)) in a setting sequence U for (v, w) . Recall that the arc with range 1 is (a_1, b_1) .

In the case $k = 2$, we have either $r = 1$ (when cases 1 or 2 in Claim 2 are used to obtain the second arc), or $s = 1$ (when cases 3 or 4 are used). When $r = 1$ we are already done. When $s = 1$, by Claim 5(b) we know that (b_1, b_2) is an old arc, and we are done.

In the general case, assume by inductive hypothesis that the claim holds for all arcs with range less than k in some setting sequence, and that the range of (v, w) (or, equivalently, of (a_s, b_r)) in U is k . We have two cases.

Case A. The arc preceding (a_s, b_r) in U is (a_s, b_{r-1}) . By inductive hypothesis, for (a_s, b_{r-1}) there is a canonical setting sequence $U^0 := (a_1, b_1), (a_2, b_1), \dots, (a_s, b_1)$ and (if $b_{r-1} \neq b_1$) (a_s, b'_2) , meaning that $b'_2 = b_{r-1}$ when b'_2 exists, and $b_1 = b_{r-1}$ when b'_2 does not exist. We have two (sub)cases:

- A.1. When b'_2 exists, we have that $U^1 := U^0.(a_s, b_r)$ (this is concatenation) is a setting sequence for (a_s, b_r) , in which (a_s, b_r) is obtained from (a_s, b'_2) using the same case of Claim 2 as used in U . Notice that the case 3 with a new arc (b'_2, b_r) cannot appear, since then in any setting sequence U' for (b'_2, b_r) with arc sources c_x and arc targets d_y , we have that $(b'_2, b_r) = (c_i, d_j)$ for some i and j , implying that (b_1, c_i) is an old arc (as $c_i = b'_2$), a contradiction with Claim 5(a). Then only case 3 with an old arc, and case 4 may occur. Both cases imply that (b_1, b_r) is an old arc, as follows. In case 3 with an old arc (b'_2, b_r) , the transitivity using the old arc (b_1, b'_2) implies indeed the construction of (b_1, b_r) in step 8 of Algorithm 1. If $\{b'_2, b_r\}$ is a local basis (i.e. case 4 is used), we deduce that b_1 is a top for it, by Claim 4(b). Now, since (b_1, b'_2) is an old arc by inductive hypothesis, we deduce that (b_1, b_r) is also an old arc obtained from the NB-constraint with top b_1 and basis $\{b'_2, b_r\}$. Thus (b_1, b_r) is an old arc in all cases. Then $U^2 = (a_1, b_1), \dots, (a_s, b_1), (a_s, b_r)$ is a setting sequence for (a_s, b_r) , which is canonical if we ensure that b_r is distinct from all a_i , $1 \leq i \leq s$. This is guaranteed by Claim 5(a).
- A.2. When b'_2 does not exist, we have that $U^1 := U^0.(a_s, b_r)$ is a canonical setting sequence for (a_s, b_r) . Indeed, as $b_1 = b_{r-1}$ we know that (a_s, b_r) is obtained from (a_s, b_1) using case 3 or 4 in Claim 2. Moreover, by Claim 4(b), b_1 belongs to no basis, thus (b_1, b_2) is an old or new arc. But the latter possibility is forbidden by Claim 5(b).

Case B. The arc preceding (a_s, b_r) in U is (a_{s-1}, b_r) . By inductive hypothesis, for (a_{s-1}, b_r) there is a canonical setting sequence $U^0 := (a_1, b_1), (a_2, b_1), \dots, (a_{s-1}, b_1)$ and (if $b_r \neq b_1$) (a_{s-1}, b'_2) , meaning that $b'_2 = b_r$ when b'_2 exists, and $b_1 = b_r$ when b'_2 does not exist. We have two (sub)cases:

- B.1. When b'_2 exists, we show that the sequence $U^1 = (a_1, b_1), \dots, (a_{s-1}, b_1), (a_s, b_1), (a_s, b'_2)$ is the sought canonical sequence. Clearly, (a_{s-1}, b_1) is obtained from (a_1, b_1) using the setting sequence U^0 from which (a_{s-1}, b_r) is useless in this case. Also, (a_s, b'_2) is obtained from (a_s, b_1) and (b_1, b'_2) by transitivity (case 3 in Claim 2). It remains to show that (a_s, b_1) is deduced from (a_{s-1}, b_1) and $\{a_{s-1}, a_s\}$. In U , $\{a_{s-1}, a_s\}$ is used to deduce (a_s, b_r) from (a_{s-1}, b_r) , using either case 1 or case 2 in Claim 2. If case 1 is used, then (a_s, a_{s-1}) is an arc (new or old), and it allows to deduce (a_s, b_1) from (a_{s-1}, b_1) using the transitivity. If case 2 is used, then $\{a_{s-1}, a_s\}$ is a local basis, thus b_1 is a top of it. The resulting NB-constraint allows to deduce (a_s, b_1) from (a_{s-1}, b_1) in this case too.

B.2 When b'_2 does not exist, we have that $b_r = b_1$ and $U^1 := U^0.(a_s, b_1)$ is a canonical setting sequence for (a_s, b_r) .

Claim 7. *Let F be a directed linear profile on $[n] \cup \{0, n+1\}$. Then the NB-transitive closure G^1 obtained in step 5 of Algorithm 3 when b_1 (respectively a_1) are chosen as in step 3 (respectively step 4) has no circuit.*

Proof. Assume a circuit $d_1, d_2, \dots, d_c, c \geq 2$, is created in G^1 . Because of the transitive closure, a shortest such circuit has length 2. Let then d_1, d_2 form a 2-circuit and assume that (at least) (d_1, d_2) is a new arc. Then, according to Claim 6, there exists a canonical setting path with vertices $a_1, \dots, a_s (= d_1)$ and $b_1, \dots, b_r (= d_2)$ ($r \in \{1, 2\}$). Consequently (d_2, d_1) cannot be an old arc, since then in G either we have directly that (b_1, d_1) is an old arc (when $r = 1$ and thus $d_2 = d_1$) or the transitivity guarantees the same conclusion when $r = 2$. But this yields a contradiction with Claim 5(a).

We deduce that (d_2, d_1) is a new arc, implying again the existence of a canonical setting path with vertices $a'_1, \dots, a'_{s'} (= d_2)$ and $b'_1, \dots, b'_{r'} (= d_1)$ ($r' \in \{1, 2\}$). But $b'_1 = b_1$ and $a'_1 = a_1$. Consequently we have either that $b_1 = d_1$ (when $r' = 1$) or that (b_1, d_1) is an old arc (when $r' = 2$). In the former case we have a contradiction with affirmation (a) in the definition of a canonical setting path since $d_1 = a_s = b_1$. In the latter case, we have again a contradiction with Claim 5(a). ■

We are now ready to prove the main theorem:

Theorem 1. DIRECTED *MinMax*-BETWEENNESS is polynomial for linear *MinMax*-profiles.

Proof. Given a linear *MinMax*-profile F , we first apply Algorithm 1 and, if it returns a pair (G, SilNB) , we apply Algorithm 3. To show the correctness of the algorithm, we show the answer is "No" iff there is no permutation whose *MinMax*-profile is F .

If the answer is "No", then Algorithm 1 returns that G is not a DAG, which occurs iff some *MinMax*-constraints from F cannot be simultaneously satisfied. Thus, there is no permutation with *MinMax*-profile F .

Now, assume there is no permutation whose *MinMax*-profile is F , and suppose by contradiction that the algorithm returns a permutation P . We show that P satisfies all the *MinMax*-constraints in F , yielding a contradiction with the hypothesis. The permutation P is output at the end of Algorithm 3, showing that Algorithm 1 finishes with a DAG G . Then, in Algorithm 3 every execution of the **while** loop in steps 1-6 satisfies at least one silent NB-constraint and, by Claim 7, creates no circuit. Therefore, the pair (G, SilNB) obtained at the end of each execution consists again in a DAG G with R -, B -, T - and NB -arcs, and a set SilNB with smaller size than the previous one. Thus the **while** loop ends when $\text{SilNB} = \emptyset$ and yields a DAG G that satisfies all the constraints imposed by the *MinMax*-profile F . Any topological order of G , including P , is thus a permutation with *MinMax*-profile F . The hypothesis that there is no permutation with *MinMax*-profile F is thus contradicted.

The execution time of the algorithm is clearly dominated by the $|\text{SilNB}|$ computations of the NB-transitivity closure in step 5 of Algorithm 3. Now, the number of NB-constraints in SilNB is in $O(n^2)$ (we have at most one NB-constraint $\neg(t \xrightarrow{a} t+1)$ for each t and each a) and the NB-transitivity closure is clearly performed in polynomial time, thus the resulting algorithm is polynomial. ■

5 Generalizations

In this section, we generalize the definition of *MinMax*-profiles so as to allow them to carry different amounts of information, depending on an integer parameter k .

Definition 3. Let k be a positive integer with $1 \leq k \leq n+1$. The k -profile of a permutation P on $[n] \cup \{0, n+1\}$ is the set of k -constraints

$$M_k(P) = \bigcup_{i=1}^k \{t \xrightarrow{[\min_{t,t+i}, \max_{t,t+i}]} t+i \mid 0 \leq t \leq n+1-i\}$$

where $\min_{t,t+i}$ ($\max_{t,t+i}$ respectively) is the minimum (maximum respectively) value in the interval delimited on P by the element t (included) and the element $t + i$ (included).

Note that *MinMax*-profiles as defined in Section 2 are the 1-profiles. A k -profile is thus a *MinMax*-profile augmented with longer-range information of the same nature as the *MinMax*-profile itself, for pairs $\{t, t + i\}$ with i at most equal to k . Then all the definitions related to *MinMax*-profiles generalize to k -profiles in an obvious way, allowing us to state the following variant of the *MinMax*-Betweenness Problem:

(DIRECTED OR NOT) *k-MinMax* BETWEENNESS

Input: A positive integer n , a (directed or not) k -profile F_k on $[n] \cup \{0, n + 1\}$.

Question: Is there a permutation P on $[n] \cup \{0, n + 1\}$ whose k -profile is F_k ?

Similarly to the *MinMax*-BETWEENNESS problem, the *k-MinMax* BETWEENNESS problem provides a k -profile, which imposes B-constraints and NB-constraints for the permutations associated with it, if any. The existence of at least one permutation raises the question of its uniqueness, allowing to know whether the permutation is characterized by its k -profile or not. More formally, we state the two following problems:

(DIRECTED OR NOT) *MinMax*-RECONSTRUCTION

Input: A positive integer n .

Requires: Find the minimum value of k such that (directed or not) *k-MinMax* BETWEENNESS has at most one solution, for each possible k -profile F_k on $[n] \cup \{0, n + 1\}$.

(DIRECTED OR NOT) UNIQUE *k-MinMax* BETWEENNESS

Input: A positive integer n , a (directed or not) k -profile F_k on $[n] \cup \{0, n + 1\}$.

Requires: Decide whether F_k is the k -profile of a unique permutation on $[n] \cup \{0, n + 1\}$, or not. In the positive case, find the unique permutation associated with F_k .

Problems *k-MinMax* BETWEENNESS and UNIQUE *k-MinMax* BETWEENNESS are clearly related, but do not allow easy deductions in one sense or the other. For instance, even if we have a solution for the DIRECTED *MinMax*-BETWEENNESS in the case of a linear profile (see Section 4), we know nothing about the uniqueness of the permutation P the algorithm outputs (when such a permutation exists).

In the subsequent, we solve the *MinMax*-RECONSTRUCTION problem in the undirected case, and give a lower bound for the directed case. We assume wlog that the k -profiles we use are compatible with the assumption that 0 and $n + 1$ are respectively the leftmost and the rightmost element in the permutations we are dealing with. Then we prove the following result:

Theorem 2. *The minimum k in (directed or not) MinMax-RECONSTRUCTION satisfies:*

- (a) $k = \max\{1, n - 3\}$ in *MinMax*-RECONSTRUCTION.
- (b) $k \geq \lceil \frac{n-3}{2} \rceil$ in DIRECTED *MinMax*-RECONSTRUCTION, for $n \geq 4$. For $n = 1, 2, 3$, we have $k = 1$.

The proof is based on the following claim.

Claim 8. *Let n be a positive integer, and F_k be a (directed or not) k -profile on $[n] \cup \{0, n + 1\}$ ($1 \leq k \leq n + 1$). Then:*

1. *In all the permutations whose k -profile is F_k (if any), the elements 1 and n have precisely the same positions, denoted q_1 and q_n*

2. If $l = \min\{q_1, q_n\}$ and $r = \max\{q_1, q_n\}$, then the sets X, Y and Z of elements situated respectively strictly between the positions 0 and l (for X), l and r (for Y), r and $n + 1$ (for Z) are the same over all the permutations with k -profile F_k (if any).

Proof. Assuming at least one permutation corresponding to F_k exists, let P be such a permutation. Denote q_1 the position of 1 on P and successively consider the B-constraints

$$n \xleftrightarrow{m_{n,n+1}} n+1, n-1 \xleftrightarrow{m_{n-1,n}} n, \dots, 2 \xleftrightarrow{m_{2,3}} 3.$$

The first B-constraint places n on the left of 1 iff $m_{n,n+1} = 1$, the second one places $n-1$ on the opposite side of 1 with respect to n iff $m_{n-1,1} = 1$ and so on. Each element in $\{n, n-1, \dots, 2\}$ is deterministically placed on the left or on the right of 1 depending only on those B-constraints. As a consequence, 1 is at the same place q_1 in all permutations corresponding to F_k .

A similar reasoning may be done with the element n and the B-constraints:

$$0 \xleftrightarrow{M_{0,1}} 1, 1 \xleftrightarrow{M_{1,2}} 2, \dots, n-2 \xleftrightarrow{M_{n-2,n-1}} n-1.$$

We similarly deduce that n is at the same place q_n in all permutations corresponding to F_k , and the sets of elements situated respectively on its left and right are the same in all permutations.

Putting together the previous deductions, whatever the order of q_1 and q_n , we have that - on the one hand - $X \cup Y$ and Z are identical in all permutations, and - on the other hand - X and $Y \cup Z$ are identical in all permutations. The conclusion follows. ■

Proof of Theorem 2. We now prove affirmations (a) and (b).

Proof of affirmation (a). For $n = 1$, it is trivial. When $n \in \{1, 2, 3\}$ it is easy to prove, using Claim 8, that the 1-profile guarantees the uniqueness of the associated permutation. When $n \geq 4$, assume by contradiction that $k < n - 3$ and let P be a permutation on $[n] \cup \{0, n+1\}$ whose elements in positions 1 to 4 are $p_1 = k+2, p_2 = k+3, p_3 = 1$ and $p_4 = n$. Let F_k be the k -profile of P . According to Claim 8, the elements 1 and n are situated respectively at positions 3 and 4 in all permutations associated with F_k , and positions 1 and 2 are occupied (whatever the order) by the elements $k+2$ and $k+3$. Now, in F_k the k -constraints involving one of the elements $k+2$ and $k+3$ and another element following 1 on its right are useless for fixing the places of $k+2$ and $k+3$ since these constraints have the minimum and maximum element 1 and n . The only possibly useful k -constraints are those involving 0, 1, $k+2$ and $k+3$, but these integers have pairwise difference larger than k except for $k+2$ and $k+3$. Now, $k+2$ and $k+3$ are involved in the 1-constraint $k+2 \xleftrightarrow{[k+2, k+3]} k+3$, which does not fix them on the places 1 and 2 of the permutation. Thus, there are at least two permutations with k -profile F_k , a contradiction. We thus have $k \geq n - 3$.

We now show that if $k = n - 3$, then there is at most one permutation on $[n]$ whose k -profile is F_k . This is shown by induction on n .

When $n = 4$ and $k = 1$, Claim 8 guarantees that, if at least one permutation with the given 1-profile exists, then 1 and 4 have fixed places, and 2 (respectively 3) is located in the same set among X, Y, Z in all suitable permutations. If 2 and 3 are in different sets, then the uniqueness is guaranteed. Otherwise, either 2 and 3 are in a set delimited by the position of 1, and then the constraint $1 \xleftrightarrow{[m_{1,2}, M_{1,2}]} 2$ allows to deduce whether 3 separates 1 and 2 or not (thus fixing the positions of 2 and 3), or they are in a set delimited by $n (= 4)$, and then the constraint $3 \xleftrightarrow{[m_{3,4}, M_{3,4}]} 4$ allows to deduce whether 2 separates 3 and 4 or not. In all cases, all the elements are located at fixed places, thus the permutation associated with the 1-profile is unique.

Assume now, by inductive hypothesis, that for all $n' < n$, a $(n' - 3)$ -profile either has no associated permutation, or has exactly one. Let now F_{n-3} be a $(n-3)$ -profile for permutations on $[n] \cup \{0, n+1\}$, and let q_1, q_n, l, r, X, Y, Z be defined according to Claim 8, assuming at least one permutation exists. Denote P any of these permutations, extended with 0 and $n+1$. Let $W = X \cup \{0, n\}$, if $q_n < q_1$, and $W = X \cup Y \cup \{0, 1, n\}$, otherwise. We show that:

The elements in W have fixed places in any permutation P (3)

Note that $P[0..q_n]$, whose element set is W , is a subpermutation of P delimited by 0 and n , which are respectively the minimum and maximum element in $P[0..q_n]$. Now, renumber the elements of $P[0..q_n]$ from 0 to $n' + 1$ according to their increasing values, where $n' < n$ and $n' + 1$ is at position q_n . Then the resulting permutation is a permutation P' on $[n']$ augmented with 0 and $n' + 1$.

Denote $F_{n'-3}$ the $(n' - 3)$ -profile of this permutation, and let us show that P' is unique. For $n' \geq 4$, we show that when F_{n-3} is known, $F_{n'-3}$ is also known, and then apply inductive hypothesis to deduce that $F_{n'-3}$ (and thus F_{n-3}) fixes the places of the elements in $P[0..q_n]$. Whereas for $n' = 2, 3$ we show that there are enough 1-constraints deduced from F_{n-3} to guarantee the uniqueness of P' . The case $n' = 1$ is trivial.

Let $h \xrightarrow{[m_{h,h+i}, M_{h,h+i}]} h + i$ be a constraint on P' , which belongs to $F_{n'-3}$ if $n' \geq 4$ (i.e. $1 \leq i \leq n' - 3$) and to F_1 if $n' = 2, 3$. Let $b(h), b(m_{h,h+i}), b(M_{h,h+i})$ and $b(h + i)$ be respectively the labels of $h, m_{h,h+i}, M_{h,h+i}, h + i$ before renumbering. Then the difference between the labels of h and $h + i$ in the initial P satisfies:

$$b(h + i) - b(h) \leq (h + i) - h + (n - n' - 1). \quad (4)$$

Indeed, if x elements of P are between n and $n + 1$, then the total number of elements in P is, on the one hand, $n + 2$ (the cardinality of $[n] \cup \{0, n + 1\}$) and, on the other hand, $1 + n' + 1 + x + 1$ (given by the cardinality of W , by x and by the element $n + 1$). Then $x = n - n' - 1$, and it represents the maximum number of elements that can miss between $b(h + i)$ and $b(h)$, additionally to the values separating them in P' , i.e. $(h + i) - h$. But then from equation (4) we deduce:

$$b(h + i) - b(h) \leq i + n - n' - 1. \quad (5)$$

Case $n' \geq 4$. From equation (5) we deduce with $i \leq n' - 3$ that $b(h + i) - b(h) \leq n' - 3 + n - n' - 1 \leq n - 4$, meaning that $b(h) \xrightarrow{[b(m_{h,h+i}), b(M_{h,h+i})]} b(h + i)$ is a constraint from F_{n-3} , yielding the constraint $h \xrightarrow{[m_{h,h+i}, M_{h,h+i}]} h + i$ of $F_{n'-3}$ after renumbering. Of course, this affirmation is true since the renumbering keeps the order between the elements, and thus the (renumbered) minimum and maximum value of each given interval. Thus $F_{n'-3}$ is deducible from F_{n-3} and, by inductive hypothesis, the permutation P' is uniquely determined by $F_{n'-3}$.

Case $n' = 2$. Then, as assumed above, $i = 1$ and thus equation (5) implies $b(h + 1) - b(h) \leq 1 + n - 2 - 1 = n - 2$ which is larger than $n - 3$. This shows that all the 1-constraints on P' with $b(h + 1) - b(h) \leq n - 3$ are deducible from constraints in F_{n-3} but the 1-constraints on P' with $b(h + 1) - b(h) = n - 2$ are not. These latter 1-constraints are obtained when $b(h + 1) - b(h) = 1 + n - 2 - 1$ (according to equation (5)), that is, when $b(h + 1) - b(h) = n - 2$. To achieve this with 0, n and the other two elements e, f in W (w.l.o.g. assume $e \leq f$) we must have either $e = n - 2$, and thus $f = n - 1$, (such that $e - 0 = n - 2$), or $e = 1$ and $f = n - 1$ (such that $f - e = n - 2$), or $e = 1$ and $f = 2$ (such that $n - f = n - 2$). In all cases, exactly one 1-constraint is missing (i.e. not resulting from F_{n-3}) but the uniqueness of the permutation is still guaranteed, since the two other 1-constraints are sufficient to fix the elements in a 4-permutation (including the endpoints 0 and 4).

Case $n' = 3$. Using (5) and the information that $i = 1$, we deduce that $b(h + 1) - b(h) \leq 1 + n - 3 - 1 = n - 3$, and thus all the 1-constraints are available for P' . As the theorem is true for permutations on 3 elements, then we are done.

Affirmation (3) is proved. Similarly, we show that the places of the elements situated on each permutation P between the element 1 (in position q_1) and the element $n + 1$ are fixed. Thus, all the elements of each permutation P are in fixed places, and there is only one permutation P with the $(n - 3)$ -profile F_{n-3} .

Proof of affirmation (b). Similarly to the undirected case, in the directed case assume by contradiction that $k < \lceil \frac{n-3}{2} \rceil$ and build P as in the undirected case, but with $p_2 = 2k + 3$ instead of $p_2 = k + 3$ (thus

avoiding the directed k -constraint $k + 2 \xrightarrow{[k+2, k+3]} k + 3$ or $k + 2 \xleftarrow{[k+2, k+3]} k + 3$, which fixes in the directed case the positions of $k + 2$ and $k + 3$). Then, with the directed k -profile F_k of P , no k -constraint exists involving $0, 1, k + 2, 2k + 3$, and thus $k + 2$ and $2k + 3$ may permute on the positions 1 and 2. Therefore, at least two permutations exist with the k -profile F_k , a contradiction.

Thus the uniqueness of the permutation implies $k \geq \lceil \frac{n-3}{2} \rceil$. ■

6 Conclusions and Perspectives

In this paper, we investigated some problems related to the construction of a permutation from a *MinMax*-profile or, more generally, from some k -profile, with $1 \leq k \leq n$. For the first of these problems, the *MinMax*-BETWEENNESS problem, we noticed the main difficulties of the directed version and gave a polynomial particular case.

The undirected version is even more difficult, due to differences with respect to the directed version that we present hereafter. First, as the relative position of t and $t + 1$ (*i.e.* the arc of G between t and $t + 1$) is not directly given by the *MinMax*-profile, the B-constraints cannot be directly exploited as in steps 3-5 of Algorithm 1. The construction of those two types of arcs, the R -arcs and the B -arcs, must therefore be integrated into the Build-Closure algorithm, where the B-constraints must be considered as well as the NB-constraints when seeking new arcs to be added to G . It may be noticed that, similarly to the case of the NB-constraints, any of the B-constraints $t \xrightarrow{m_t} t + 1, t \xleftarrow{M_t} t + 1$ has two possible settings, resulting either in the set of new arcs $Arcs^+ = \{(t, t + 1), (t, m_t), (t, M_t), (m_t, t + 1), (M_t, t + 1)\}$, or in the set of new arcs $Arcs^- = \{(t + 1, t), (m_t, t), (M_t, t), (t + 1, m_t), (t + 1, M_t)\}$. When one arc is set, then the four other arcs are set accordingly. When no arc is set, the B-constraints are silent. The algorithm obtained from Algorithm 1 by performing the indicated changes thus outputs either the answer No, or G and two sets $SilNB$ and $SilB$ of silent NB- and silent B-constraints respectively. We thus arrive at the second main difference between the directed and undirected case. Any setting sequence must allow to deduce new arcs also using the B-constraints, thus adding cases to those already in Claim 2, and yielding the study of the arc propagation in G even more complicated than in the directed case.

For both versions, and also for the more general k -*MinMax* BETWEENNESS problem, the algorithmic difficulty of the problem is an open problem. The same holds for the DIRECTED *MinMax*-RECONSTRUCTION problem. Also, being able to recognize a k -profile allowing to reconstruct exactly one permutation, *i.e.* solving (Directed or not) UNIQUE k -*MinMax* BETWEENNESS, would allow to identify a subclass of permutations perfectly represented by their k -profile.

References

- [1] Walter Guttmann and Markus Maucher. Variations on an ordering theme with constraints. In *Fourth IFIP International Conference on Theoretical Computer Science-TCS 2006*, pages 77–90. Springer, 2006.
- [2] Jaroslav Opatrny. Total ordering problem. *SIAM Journal on Computing*, 8(1):111–114, 1979.
- [3] Irena Rusu. MinMax-Profiles: A unifying view of common intervals, nested common intervals and conserved intervals of K permutations. *Theoretical Computer Science*, 543:90–111, 2014.